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On the optimal selection of portfolios under limited diversification

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Abstract

We address the problem of selecting portfolios which maximize the ratio of the average excess return to the standard deviation, among all those portfolios which comprise at most a pre-specified number, k, of securities. Under the assumptions of constant pairwise correlations and no short-selling, we argue that the simple ranking procedure of Elton, Gruber, and Padberg effectively solves the problem for *all* values of k, and that as a function of k, the optimal ratio increases at a *decreasing* rate. We also clarify why further generalization or extension of our results to other situations is improbable. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One basic implication of modern portfolio theory is that investors hold well-diversified portfolios. However, there is empirical evidence that individual investors typically hold only a small number of securities. [1] Market imperfections such as fixed transaction costs provide one explanation for the prevalence of undiversified portfolios. Also, a small investor who chooses to invest in only a limited number of securities can devote more attention to the individual behavior of those securities and their mean–variance characteristics. In fact, there is evidence (e.g., Evans and Archer, 1968; Fisher and Lorie, 1970; Jacob, 1974) that diversification beyond 8–10 securities may not be worthwhile provided these securities are chosen not randomly but through a systematic, optimum-seeking procedure. Citing Szegö (1980), Sengupta and Sfeir (1985) observe that the variance–covariance matrix of the returns on the securities in a portfolio that has a large number of securities tends to conceal significant singularities or near-singularities. They suggest that it may therefore be superfluous to enlarge the number of securities in a portfolio beyond a limit.

The above discussion underscores the importance of the general problem of selecting mean–variance efficient portfolios under limited diversification, i.e., an upper limit on the number of securities in the portfolio. Nevertheless, it has not received much attention in the literature. Mao (1970) and Jacob (1974) formally address the problem but develop their selection procedures under somewhat restrictive assumptions and rather high degrees of approximation. [2]

For the problem of determining mean–variance efficient portfolios under the single index model (Sharpe, 1963) of security returns and an upper limit on the number of stocks, Faaland (1974) develops an algorithm based on integer programming, which is bettered by the implicit enumeration algorithm of Blog et al. (1983). Cooper and Farhangian (1982) develop a dynamic programming approach for an extension of this problem that incorporates fixed costs of transaction.

[1] Conine et al. (1989, footnote 2, pp. 1004–1005) cite several studies that provide empirical evidence that the majority of individual investors in the U.S. hold highly undiversified portfolios. Bark (1991) states that one reason for the inadequacy of the Sharpe–Lintner–Mossin capital asset pricing model in the Korean stock market is that the portfolios of Korean investors are also highly undiversified.

[2] For instance, to compute the number of securities that optimally trades-off diversification against (fixed) transaction costs, Mao assumes that both the average excess return over the riskless rate and the standard deviation of the return are the same for all the securities in the portfolio. Further, for selecting the best portfolio among those that comprise a pre-specified number of securities, he assumes that for all of these portfolios, the nonsystematic risk is fully diversified away. Jacob assumes that the weights for the securities in the investor's portfolio are all equal to each other. She also linearizes portfolio risk in terms of the weights.

Assuming that the capital asset pricing model (Sharpe, 1964; Lintner, 1965; Mossin, 1966) holds, Brennan (1975) presents an algorithm for determining the optimal number of securities under fixed transaction costs. However, the validity of that assumption in the presence of fixed transaction costs has been questioned (Patel and Subrahmanyam, 1982). Patel and Subrahmanyam (1982) develop an efficient algorithm for the problem under the assumption that the correlation coefficient is the same for all pairs of securities (Elton and Gruber, 1995, pp. 168–169). Aneja et al. (1989) show how the average pairwise correlation coefficient can be efficiently estimated using a portfolio approach.

We are not aware of more recent algorithms for the problem or any of its variants. Nevertheless, simple ranking procedures akin to the well-known EGP algorithms (Elton et al., 1976, 1977, 1978) continue to be developed for selecting mean–variance efficient portfolios in *other* contexts such as restricted short-selling (Alexander, 1993, 1995) and institutional norms for short-selling (Kwan, 1995).

In this note, we consider the problem of selecting portfolios which maximize the ratio of the average excess return over the riskless rate to the standard deviation, among all those portfolios which comprise at most a pre-specified number, k, of securities from among the n securities that comprise the universe. We define a k-optimal portfolio as one that maximizes the ratio of the average excess return to the standard deviation over all portfolios that comprise at most k securities $(1 \leqslant k \leqslant n)$. Then, it is clear that the well-known EGP algorithms essentially find only the n-optimal portfolio for each of various cases (viz., short-selling/no short-selling and single-index/constant correlations, etc.).

The theory of computational complexity implies that the problem of finding the k-optimal portfolios for *all* the values of k (i.e., from 1 through to n) is unlikely to be efficiently solvable under the single-index model of stock returns (Blog et al., 1983). However, we formally argue that under the assumptions of constant pairwise correlations and no short-selling, the associated EGP algorithm, with a very minor modification, solves the problem for *all* values of k, and *not just* for $k = n$. Thus, we show that under the assumption of no short-selling and under minimal and plausible [3] assumptions about the variance–covariance matrix of security returns, it is extremely simple to select portfolios that are *provably optimal* among all those that comprise at most a given number, k, of securities from the universe.

We also establish that as a function of k, the optimal ratio of the average excess return to the standard deviation increases at a *decreasing* rate. Thus, we provide yet another analytical argument that the marginal benefit from diversification decreases with the number of securities in the portfolio. We

[3] The reader is referred to the discussion by Elton and Gruber (1995, pp. 168–169) of the empirical validation of the constant correlation assumption.

venture that this argument is stronger than those in the published literature because it is based on portfolios that are *optimal* under a varying upper limit on the number of securities and under minimal assumptions about the correlation structure of security returns.

The problem of determining the optimal weights in a portfolio that comprises a *given subset* of securities in the universe can be easily solved for a variety of situations (see Elton and Gruber, 1995). However, the problem of determining an optimal portfolio (and associated weights) that comprises *at most a given number* of securities from the universe is generally much harder – that is why we believe the above results are noteworthy. Thus, under the single-index model of stock returns, the former problem is easily solved by a simple ranking (EGP) algorithm, while as mentioned earlier, the theory of computational complexity suggests that an efficient algorithm for the latter problem is unlikely.

The note is organized as follows. In Section 2, we model the problem formally. In Section 3, we present the main results. Section 4 illustrates our findings on an example. Our conclusions are contained in Section 5.

2. The model

We use the following notation:

n	the number of securities in the universe.
N	the set of securities in the universe; $N = \{1, \ldots, n\}$.
k	a pre-specified upper limit on the number of securities in the portfolio $(1 \leqslant k \leqslant n)$.
x_i	the weight of security i, $i = 1, \ldots, n$ (for all i, $x_i \geqslant 0$).
R	the rate of return on the riskless asset.
R_i	the expected rate of return on security i, $i = 1, \ldots, n$.
s_i	the standard deviation of the rate of return on security i, $i = 1, \ldots, n$.
b_i	the ratio of the average excess return to the standard deviation of security i, $i = 1, \ldots, n$; thus, $b_i = (R_i - R)/s_i$.
p	an estimate of the average correlation coefficient of any pair of security returns (we assume that p is non-negative).

For their algorithm, Elton et al. (1976) do not require p to be non-negative. However, for the results of this paper, such an assumption is needed. Below, we defend this assumption.

Let s^{eq} denote the standard deviation of the return on an equally-weighted portfolio that comprises securities, $1, \ldots, m$. For the sake of simplicity, *and only for the sake of the present argument*, assume that $s_j = s$ for all j in N. Then, in the manner of Elton and Gruber (1995, p. 60), we can show that

$$(s^{\mathrm{eq}})^2 = \left(\frac{1}{m}\right)s^2 + \left(\frac{m-1}{m}\right)ps^2 = \left(\frac{1+(m-1)p}{m}\right)s^2. \tag{1}$$

Eq. (1) implies that if p is negative, then for sufficiently large values of m, $(s^{\mathrm{eq}})^2$ would become negative, which is absurd! For this reason, we assert that the assumption of non-negativity for p is mild.

Under the assumptions of constant correlation and no short-selling, the investor's problem may be formulated as follows:

$$\text{maximize } \frac{\sum_{i=1}^{n}(R_i - R)x_i}{\sqrt{\sum_{i=1}^{n} s_i^2 x_i^2 + p\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} s_i s_j x_i x_j}}$$

such that

$$x_i \geqslant 0 \quad \text{for } i = 1,\ldots,n, \text{ and at most } k \text{ of } \{x_i : i = 1,\ldots,n\} \text{ are positive.}$$

Let F be an *arbitrary* subset of N, and let $w(F)$ denote the maximum ratio of the average excess return to the standard deviation that is realizable by a portfolio which comprises *only* securities in F; formally, $w(F)$ is the maximum of

$$\frac{\sum_{i\in F}(R_i - R)x_i}{\sqrt{\sum_{i\in F} s_i^2 x_i^2 + p\sum_{i\in F}\sum_{\substack{j\in F\\ j\neq i}} s_i s_j x_i x_j}}$$

such that

$$x_i \geqslant 0 \quad \text{for all } i \text{ in } F.$$

If $(\bullet)$ is a subset of N, then we let $|(\bullet)|$ denote the cardinality of $(\bullet)$. Then, the problem of our interest may be reformulated as

$$\boldsymbol{P}: \text{ maximize } w(F)$$

$$\text{over all } F \subset N \text{ such that } |F| \leqslant k.$$

3. The results

Our algorithm for solving $\boldsymbol{P}$ and the arguments for its validity are based on a procedure, GET-PORTFOLIO. If F denotes an arbitrary subset of N, then GET-PORTFOLIO returns a portfolio comprising a subset of securities from F; we refer to this subset as S_F.

Without loss of generality, we assume the securities in the universe are numbered in descending order of $\{b_i\}$; thus, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. If F denotes an arbitrary subset of N, then for $t = 1, \ldots, |F|$, let $i(t;F)$ denote the security with the tth greatest value of b among the securities in F. (We can always define $\{i(t;F) : t = 1, \ldots, |F|\}$ such that $i(1;F) < i(2;F) < \cdots < i(|F|;F)$.)

3.1. GET-PORTFOLIO

Input: F (an arbitrary subset of N).

Output: A portfolio comprising a subset of securities from F, namely, S_F.

1. *If* $b_{i(1;F)} \leqslant 0$, *then* set $t = 0$ and go to step 4; *else*, initialize t to 1.
2. *If* $t \geqslant |F|$ or $b_{i(t+1;F)} \leqslant p(\sum_{u=1}^{t} b_{i(u;F)})/((t-1)p+1)$, *then* go to step 4; *else*, increment t by 1.
3. Go to step 2.
4. Set $S_F = \{i(u;F) : u = 1, \ldots, t\}$.

Construct the weights $\{x_i : i \in F\}$ as follows:

$$\text{for } u = 1, \ldots t,\ x_{i(u;F)} \text{ is in proportion to } \left(\frac{1}{s_{i(u;F)}}\right)\left(b_{i(u;F)} - \frac{p\sum_{v=1}^{t} b_{i(v;F)}}{(t-1)p+1}\right); \tag{2}$$

$$\text{for } u = t+1, \ldots, |F|,\ x_{i(u;F)} := 0. \tag{3}$$

The following proposition validates GET-PORTFOLIO.

Proposition 1. *If F denotes an arbitrary subset of N, then*

$$w(F) := \sqrt{\left(\frac{1}{1-p}\right)\left(\sum_{i\in S_F} b_i^2 - \frac{p\left(\sum\limits_{i\in S_F} b_i\right)^2}{p(|S_F|-1)+1}\right)}. \tag{4}$$

Further, the portfolio that attains $w(F)$ *is given by Eqs.* (2) *and* (3).

The reader is referred to Elton and Gruber (1995, pp. 205–206) for a proof of Proposition 1; when $F = N$, GET-PORTFOLIO is precisely the EGP algorithm that is associated with the constant correlation assumption and no short-selling (Elton et al., 1976; Elton and Gruber, 1995, pp. 195–197). As Elton and Gruber (1995) note, step 2 in the procedure represents the search for the optimal *cut-off* value for the ratio of excess return to standard deviation. Thus, those securities in F with a ratio that is greater than the cut-off have positive weights, while those securities in F with a ratio that is not greater than the cut-off have zero weights.

If $b_i \leqslant 0$ for all $i = 1 \ldots n$, then it is optimal to invest in none of the n securities. Hence, to avoid trivialities, in the rest of the note, we assume that $b_1 > 0$.

The following proposition is the key result that underlies our algorithm for P.

Proposition 2. *Let F be a subset of N containing m $(2 \leqslant m \leqslant n)$ securities such that $S_F = F$. Let L denote the largest-numbered security in F. (Thus, L has the smallest value of b among all the securities in F.) Let j be a security which is not in F such that $j < L$. Then, $w(F \cup \{j\} \setminus \{L\}) \geqslant w(F)$.*

The proofs of this proposition and of the rest of the results of the note (except Proposition 4) are presented in Sankaran and Patil (1998).

Proposition 2 has an important corollary. We recall to the reader that a k-optimal portfolio is defined as one which maximizes the ratio of the average excess return to the standard deviation, over all portfolios that comprise at most k securities $(k \geqslant 2)$.

Corollary 3. *There is a k-optimal portfolio which comprises securities $\{1, \ldots, s\}$ for some $s \leqslant k$. Further, GET-PORTFOLIO finds such a portfolio when F is defined as $\{1, \ldots, k\}$.*

Corollary 3 implies that the following algorithm finds the k-optimal portfolio for all values of k.

1. Initialize k as 2. The 1-optimal portfolio comprises only security 1.
2. *If* $b_k \leqslant p(\sum_{j=1}^{k-1} b_j/(k-2)p+1)$, *then* go to step 4.
3. The k-optimal portfolio comprises securities 1 to k and for $i = 1, \ldots, k$, the optimal weight of security i is proportional to $(1/s_i)(b_i - p(\sum_{j=1}^{k} b_j)/((k-1)p+1))$. Increment k by 1. If $k \leqslant n$, go to step 2.
4. Set K as $k-1$ and stop; for all $k > K$, the k-optimal portfolio is identical to the K-optimal portfolio.

Note that the above procedure is the same as the EGP algorithm except for the explicit computation of the optimal weights in step 3 for each value of k. Thus, although the EGP algorithm is just the same as setting $F = N$ and executing GET-PORTFOLIO, *in the process of execution*, it *effectively* finds the k-optimal portfolio for *all* values of k.

Now, for $k = 1, \ldots, K$, let $W(k)$ denote the ratio of the average excess return to the standard deviation of the k-optimal portfolio; then

$$W(k) = \sqrt{\left(\frac{1}{1-p}\right)\left(\sum_{i=1}^{k} b_i^2 - \frac{p\left(\sum_{i=1}^{k} b_i\right)^2}{(k-1)p+1}\right)}.$$

The following proposition implies that the marginal benefit from diversification decreases with the number of securities in the portfolio. It is proved in the appendix.

Proposition 4. *For* $k = 2, \ldots, K - 1$, $W(k) - W(k-1) > W(k+1) - W(k)$.

4. An example

We illustrate our results on an abridged version of the example that Elton and Gruber (1995) use to illustrate the EGP algorithm for the constant correlation case. We assume that $p = 0.5$. Table 1 presents the rest of the data and Table 2 presents the results of the algorithm. As described on p. 196 of Elton and Gruber (1995), $K = 3$. Hence, for $k > 3$, the k-optimal portfolio is the same as the 3-optimal portfolio.

Through Table 2, we note that the marginal benefit of adding a second security, which is 0.718, is greater than the marginal benefit of adding a third security, namely 0.085 – this is only to be expected owing to Proposition 4.

5. Discussion and conclusion

In this note, we have addressed the problem of selecting mean–variance efficient portfolios under limited diversification. Specifically, under the

Table 1
Data to illustrate the algorithm for finding the k-optimal portfolio for all values of k

Security number	Excess return	Standard deviation	Excess return to standard deviation
1	24	3	8.0
2	14	2	7.0
3	24	4	6.0
4	30	6	5.0
5	9	2	4.5

Table 2
The various k-optimal portfolios[a]

k	Securities in the k-optimal portfolio	Optimal weights of the securities in the k-optimal portfolio	Excess return to standard deviation in the k-optimal portfolio
1	1	1	8
2	1, 2	0.5, 0.5	8.718
3	1, 2, 3	0.463, 0.442, 0.095	8.803

[a] For $k > 3$, the k-optimal portfolio is the same as the 3-optimal portfolio.

assumptions of constant pairwise correlations and no short-selling, we have examined the problem of selecting portfolios which maximize the ratio of the average excess return to standard deviation, among all those portfolios which comprise at most a given number, k, of securities from the universe. We have formally demonstrated that the associated EGP algorithm effectively solves the problem for *all* values of k, and that the marginal benefit from diversification decreases with the number of securities.

Given that an efficient algorithm for the problem is unlikely under the single-index model of stock returns, one is naturally tempted to examine whether the results of this note extend to the case when all pairwise correlations are constant and short-selling *is* allowed. As it turns out, one can follow the reasoning of Patel and Subrahmanyam (1982, Theorem 1) and show that in this case, the optimal set of securities for investment in a k-optimal portfolio is of the form $\{1, 2, \ldots, u_k - 1, u_k, l_k, l_k + 1, \ldots, n\}$ where $u_k + n - l_k + 1 = k$. [4]

In the case when all pairwise correlations are the same and short-selling is *not* allowed, the results are very strong largely because the family of k-optimal portfolios is nested, i.e., for all k, the set of securities in a $(k+1)$-optimal portfolio contains those in a k-optimal portfolio. It would appear from Theorem 2 of Patel and Subrahmanyam (1982) that this is also true when all pairwise correlations are constant and short-selling *is* allowed, namely, that for all k, *either* $u_{k+1} = u_k + 1$ (and $l_{k+1} = l_k$) *or* $u_{k+1} = u_k$ (and $l_{k+1} = l_k - 1$). Unfortunately, however, the proof of that theorem is invalidated by a very slight algebraic error; in the last inequality on p. 308 of Patel and Subrahmanyam (1982), the term "(m−2)" is incorrectly used in place of "(m−1)".

The upshot of the above discussion is that when pairwise correlations are constant and short-selling is allowed, for given k, we can still find the k-optimal portfolio by first evaluating the maximum ratio of excess return to standard deviation for each set of securities of the form $\{1, \ldots, u, u + n - k + 1, \ldots, n\}$ (where $0 \leqslant u \leqslant k$) and then picking the best of these $k+1$ sets. However, unlike the case when short-selling is disallowed, we can neither claim that the EGP algorithm effectively solves the problem for all values of k nor that the marginal benefit from diversification decreases with the number of securities.

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[4] Note that when securities can be sold short, we need not distinguish between F and S_F as is required when short-selling is disallowed; hence, the case when short-selling is allowed is easier to analyze.

Appendix A

Proof of Proposition 4. For $k = 1 \ldots K$, let

$$V(k) = \left(\sum_{i=1}^{k} b_i^2 - \frac{p.\left(\sum_{i=1}^{k} b_i \right)^2}{(k-1).p+1} \right).$$

We first show that for $k = 2, \ldots, K$, $V(k) > V(k-1)$, and that for $k = 2, \ldots, K-1$, $V(k) - V(k-1) \geqslant V(k+1) - V(k)$. Note that for k such that $K \geqslant k \geqslant 2$,

$$V(k) - V(k-1) = \left(\frac{(k-2)p+1}{(k-1)p+1} \right) b_k^2 + \frac{p^2.\left(\sum_{i=1}^{k-1} b_i \right)^2}{((k-1)p+1).((k-2)p+1)} - \frac{2pb_k\left(\sum_{i=1}^{k-1} b_i \right)}{(k-1)p+1},$$

i.e.

$$V(k) - V(k-1) = \left(\frac{(k-1)p+1}{(k-2)p+1} \right). \left[\frac{(k-2)p+1}{(k-1)p+1} \left(b_k - \frac{p\left(\sum_{i=1}^{k-1} b_i \right)}{(k-2)p+1} \right) \right]^2. \tag{A.1}$$

From the definition of the algorithm for selecting a k-optimal portfolio, it follows that the right-hand side of Eq. (A.1) is positive; hence, for $k = 2, \ldots, K$, $V(k) > V(k-1)$.

We will now show that for $k = 2, \ldots, K-1$,

$$\frac{(k-2)p+1}{(k-1)p+1} \left(b_k - \frac{p\left(\sum_{i=1}^{k-1} b_i \right)}{(k-2)p+1} \right) \geqslant \frac{(k-1)p+1}{kp+1} \left(b_{k+1} - \frac{p\left(\sum_{i=1}^{k} b_i \right)}{(k-1)p+1} \right). \tag{A.2}$$

We can rewrite the difference between the left-hand and right-hand sides of (A.2) as

$$\frac{(k-1)p+1}{kp+1}(b_k - b_{k+1}) + \left(\frac{(k-2)p+1}{(k-1)p+1} - \frac{(k-1)p+1}{kp+1} \right) b_k + \frac{pb_k}{kp+1} - \frac{p^2.\left(\sum_{i=1}^{k-1} b_i \right)}{((k-1)p+1)(kp+1)}$$

which simplifies to

$$\begin{aligned}&\frac{(k-1)p+1}{kp+1}(b_k-b_{k+1})\\&+\frac{p((k-2)p+1)}{((k-1)p+1)(kp+1)}\left[b_k-\frac{p\left(\sum_{i=1}^{k-1}b_i\right)}{((k-2)p+1)}\right].\end{aligned}\tag{A.3}$$

The first term in Eq. (A.3) is non-negative because $b_k \geqslant b_{k+1}$. The second term in Eq. (A.3) is non-negative by the definition of the algorithm for selecting a k-optimal portfolio and by the non-negativity of p. Therefore, the expression (A.3) is non-negative, and we have established (A.2).

Now, since $0 \leqslant p \leqslant 1$, for all $k \geqslant 2$,

$$\frac{(k-1)p+1}{(k-2)p+1} \geqslant \frac{kp+1}{(k-1)p+1}.$$

Hence, from Eqs. (A.1) and (A.2), it follows that for $k = 2, \ldots, K-1$, $V(k) - V(k-1) \geqslant V(k+1) - V(k)$. That is,

$$\begin{aligned}&\left(\sqrt{V(k)}-\sqrt{V(k-1)}\right)\left(\sqrt{V(k)}+\sqrt{V(k-1)}\right)\\&\geqslant\left(\sqrt{V(k+1)}-\sqrt{V(k)}\right)\left(\sqrt{V(k+1)}+\sqrt{V(k)}\right).\end{aligned}\tag{A.4}$$

However, $0 < V(1) < V(2) < \cdots < V(K)$, and therefore, Eq. (A.4) implies that

$$\sqrt{V(k)}-\sqrt{V(k-1)} > \sqrt{V(k+1)}-\sqrt{V(k)}.$$

Since

$$W(k)-W(k-1)=\frac{\sqrt{V(k)}-\sqrt{V(k-1)}}{\sqrt{1-p}}$$

and

$$W(k+1)-W(k)=\frac{\sqrt{V(k+1)}-\sqrt{V(k)}}{\sqrt{1-p}},$$

the proposition stands proved. □

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